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LETTER TO THE EDITOR

q-oscillators realization of $F_q(4)$ and $G_q(3)$

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Abstract. A realization of the quantum superalgebras corresponding to F(4) and G(3) is given in terms of q-deformed bosonic and fermionic oscillators.

The quantum groups [1, 2] are actually a topic of intensive research both in mathematical physics and mathematics. Roughly speaking the q-deformed universal algebra $U_q(G)$ of a semi-simple Lie algebra G is defined by a set of q depending relations between the generators of G in the Serre presentation endowed with a Hopf algebra structure, using as input the Cartan matrix of G (see for instance [3] for a more precise definition). It is a natural idea to apply these ideas to the Lie superalgebra (SLA) [4] using as input the defining Cartan matrix of SLA [5] (for a review see [6]). Now it is well known that an explicit construction of $U_q(A_n)$ could be obtained by introducing q-analogue of the harmonic oscillator boson operator [7, 8] satisfying a q-deformed Weyl algebra. Introducing also the q-analogue of the fermionic operators satisfying a q-deformed Clifford algebra [5, 3] this construction has been generalized to the q-universal enveloping algebra of all classical Lie algebras [3], to the exceptional Lie algebras [9, 10]. (Really in [10] the construction of $U_q(G_2)$ has been obtained in terms of q-quasiparafermions (called 'q-skedofermions'); however, we give below a realization of $U_q(G_2)$ in terms of q-fermions).

A similar construction for the classical SLAs has been given in [11]. The aim of this letter is to present a similar realization of $F_q(4)$ and $G_q(3)$. A realization of F(4) and G(3) in terms of bosonic–fermionic operators has been given in [12], where, in order to express the generators of the even part of the SLAs as bilinears of fermionic operators and the generators of the odd part as bilinears in one boson and one fermion, use has been made of the embedding $G_2 \subset B_3 \subset D_4$. As a consequence a generator of B_3 (respectively G_2) is expressed as linear combination of the generators of D_4. Since a peculiar feature of the deformation is the lack of linearity in the defining relations, this realization is not suitable to obtain a realization of the corresponding q-superalgebras, at least in a straightforward way.

So we follow a slightly different approach.

Let us recall the definition of G_q associated with a simple G SLA of rank r [5, 11].

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The quantum superalgebra G_q of the universal enveloping of G is generated by $3r$ elements E_i, F_i and H_i which satisfy $(i, j = 1 = 1, \ldots, r)$

$$\begin{aligned} &[E_i, F_i\} = \delta_{ij}[H_i]_{q_i^2} \\ &[H_i, H_j] = 0 \\ &[H_i, E_j] = a_{ij} E_j \qquad [H_i, F_j] = -a_{ij} F_j \end{aligned} \tag{1}$$

where [, } is the supercommutator, (a_{ij}) is the Cartan–Kac matrix of G, $q_i = q^{d_i}$, d_i being non-zero integers with greatest common divisor equal to one such that $d_i\, a_{ij} = d_j\, a_{ji}$, and

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{2}$$

The quantum superalgebra is endowed with a Hopf algebra structure. The action of the coproduct Δ, antipode S and co-unit ε on the generators is as follows:

$$\begin{aligned} &\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \\ &\Delta(E_i) = E_i \otimes q_i^{H_i} + q_i^{-H_i} \otimes E_i \\ &\Delta(F_i) = F_i \otimes q_i^{H_i} + q_i^{-H_i} \otimes F_i \\ &S(H_i) = -H_i \\ &S(E_i) = -q_i^{a_{ii}} E_i \qquad S(F_i) = -q_i^{a_{ii}} F_i \\ &\varepsilon(H_i) = \varepsilon(E_i) = \varepsilon(F_i) = \varepsilon(F_i) = 0 \qquad \varepsilon(1) = 1. \end{aligned} \tag{3}$$

Further the generators obey the Serre relations which are most simply expressed in terms of the following rescaled generators

$$\hat{E}_i = E_i q_i^{-H_i} \qquad \hat{F}_i = \hat{F}_i q_i^{-H_i} \tag{4}$$

defining the q-analogue ad_q of the adjoint operation by

$$\mathrm{ad}_q = (\mu_{\mathrm{L}} \otimes \mu_{\mathrm{R}})(\mathrm{id} \otimes S)\Delta \tag{5}$$

where id is the identity operator and μ_{L}, μ_{R} are the left and right (graded) multiplications:

$$\mu_{\mathrm{L}}(x)y = xy \qquad \mu_{\mathrm{R}}(x)y = (-1)^{\deg x \deg y} yx.$$

The Serre relations read $(i \neq j)$

$$(\mathrm{ad}_q \hat{E}_i)^{1-\hat{a}_{ij}} \hat{E}_j = (\mathrm{ad}_q \hat{F}_i)^{1-\hat{a}_{ij}} \hat{F}_j = 0 \tag{6}$$

where $(\hat{a}_{ij})$ is the matrix obtained from (a_{ij}) by replacing the entry equal to $+1$ by -1.

In particular equation (6) can be written [3] $(\deg E_i = 0)$

$$\sum_{0 \leqslant n \leqslant 1-a_{ij}} (-1) \begin{bmatrix} 1 - a_{ij} \\ n \end{bmatrix}_{q_i^2} (E_i)^{1-a_{ij}-n} E_j (E_i)^n = 0 \tag{7}$$

where

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]q!}{[m-n]_q![n]_q} \tag{8}$$

and [11] $(\deg E_i = 1)$

$$(a_{ij} = 0) \qquad \{E_i, E_j\} = 0 \tag{9a}$$

$$(a_{ij} = 1) \qquad (E_i)^2 E_j - q_i^{-4} E_j (E_i)^2 = 0. \tag{9b}$$

Analogous equations hold, replacing E_i by F_i and q_i by q_i^{-1}.

Let us recall a few preliminary properties of q-oscillators just to establish the general framework from which the relations, that we shall use in the following, arise.

Exploiting the triality of D_4 we can introduce [13] three sets of eight fermionic oscillators which can be identified with the three fundamental representations of D_4, the adjoint representation being realized as bilinears of two fermions belonging to any of the three sets. Then the action of the adjoint on the fundamental representations requires that the three sets are not independent and, for consistency, have to satisfy a set of relations given in [13].

We recall also that a spinorial representation of D_4 still transforms under $B_3 \subset D_4$ as the spinorial representation, the highest weight (HW) and lowest weight (LW) still remaining HW (LW) in the decomposition. Moreover the spinorial representation of B_3 decomposes under $G_2 \subset B_3$ as the fundamental and the singlet representations of G_2, the HW (LW) of the spinorial still remaining the HW (LW) of the fundamental representation.

In the following we introduce only the set of relations we need explicitly in our realization.

Consider six fermionic oscillators (a_i^+, a_i) $(i = 1, 2, 3)$, which can be considered as a subset of the fermionic operators spanning the vectorial representation of a D_4, and two fermionic oscillators (ψ^+, ψ) which are assumed to transform, respectively, as the HW and LW of the spinorial representations of the same D_4. Then these fermions are not independent and they satisfy the following relations $(i = 1, 2, 3)$:

$$h_i = a_i^+ a_i$$

$$[h_i, \psi^+] = \tfrac{1}{2}\psi^+ \qquad [h_i, \psi] = -\tfrac{1}{2}\psi \tag{10a}$$

$$[a_i^+, \psi^+] = [a_i, \psi] = 0 \tag{10b}$$

$$[[a_i^+, \psi], a_j] = [[a_i, \psi^+], a_j^+] = 0 \qquad (i \neq j). \tag{10c}$$

As clearly $\psi^+\psi$ commutes with all h_i, it can be expressed (up to an additional c-number) as a linear combination of h_i:

$$\psi^+\psi = \sum_{i=1}^{3} c_i h_i. \tag{11}$$

Equation (11) allows one to compute the action of the bilinear $\psi^+\psi$ on the a_i^+, a_i operators, once the coefficients c_i have been computed. We shall do this later.

After this very short summing up, we introduce the q-oscillators we need to realize $F_q(4)$ and $G_q(3)$. We make the quantum deformation of the fermionic operators above defined, which for convenience we denote with the same notation, hoping that no confusion arises: (a_i^+, a_i, N_i), (ψ^+, ψ, Q). These q-oscillators satisfy the standard relations of a q-Clifford algebra, i.e. $(i, j = 1, 2, 3)$

$$a_i a_j^+ + q^{2\delta_{ij}} a_j^+ a_i = q^{2N_i} \tag{12a}$$

$$[N_i, a_j^+] = \delta_{ij} a_j^+ \qquad [N_i, a_j] = -\delta_{ij} a_j \tag{12b}$$

$$\{a_i, a_j\} = \{a_i^+, a_j^+\} = 0. \tag{12c}$$

The same relations hold replacing a_i^+ by ψ^+ (a_i by ψ) and N_i by Q. These sets are not mutually independent and we assume the vanishing relations (10*b*) and (10*c*) which hold for the non-deformed fermionic oscillators are preserved in the deformation and that the relations (10*a*) and (11) involving bilinears of the form $a_i^+ a_i$, $\psi^+ \psi$ still hold in the deformation replacing the bilinears respectively by N_i and Q.

We introduce a q-bosonic oscillator (b^+, b, M) satisfying a q-Weyl algebra:

$$bb^+ - q^2 b^+ b = q^{-2M} \tag{13a}$$

$$[M, b^+] = b^+ \qquad [M, b] = -b. \tag{13b}$$

The q-boson is assumed to commute with the q-fermions. Equations (12) and (13) hold replacing q by q^{-1}.

From the construction of a q-boson in terms of standard (non-deformed) bosons [14], we remark that the following properties hold: denoting by $b_i^{\pm}(\hat{q})$ a q-boson satisfying the q-Weyl algebra for the value $\hat{q}$ of the deforming parameter $(i, j = 1, 2, \ldots)$:

$$[M_i, b_j^{\pm}(q)] = \pm\delta_{ij} b_j^{\pm} \qquad \forall\, q \tag{14a}$$

$$[b_i^+(q), b_j^+(q')] = [b_i(q), b_j(q')] = 0 \qquad \forall\, q, q' \tag{14b}$$

$$[b_i^+(q), b_j(q')] = 0 \qquad i \neq j \qquad \forall\, q, q'. \tag{14c}$$

Now we make the *conjecture* that analogous relation hold for the q-fermions $a_i^+(q)$, $a_i(q)$ replacing in equations (14*b*) and (14*c*) the commutator by the anti-commutator and in equation (14*a*) M_i by N_i. We remark that this conjecture is satisfied by the standard fermions; indeed, due to the fact that the square of a number fermionic operator (i.e. h_i) is equal to the number operator itself, equations (12) are satisfied also by standard fermions [11].

We shall now provide an explicit expression of the generators of $F_q(4)$ as linears and bilinears in q-deformed fermionic operators and q-bosonic operator. The Cartan matrix of $F(4)$, in the so-called distinguished basis, is given by (see [4])

$$\begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix} \tag{15}$$

so the values of d_i are: $d_i = (2, 2, 1, -1)$.

The generators (of deg = 0) corresponding to $U_q(B_3)$ are given [3]

$$\begin{array}{ll} E_1 = a_1^+ a_2 & F_1 = a_2^+ a_1 \\ E_2 = a_2^+ a_3 & F_2 = a_2^+ a_3 \\ E_3 = a_3^+ & F_3 = a_3 . \end{array} \tag{16}$$

We have $(k = 1, 2)$

$$\begin{aligned} &[E_k,\, F_k] = [N_k - N_{k+1}]_{q^2} \\ &[E_3,\, F_3] = [2N_3 - 1]_q . \end{aligned} \tag{17}$$

Therefore

$$\begin{aligned} &H_k = N_k - N_{k+1} \qquad (k = 1,\, 2) \\ &H_3 = 2N_3 - 1 . \end{aligned} \tag{18}$$

To write the only generator of deg = 1 as bilinear in one q-fermion and one q-boson we define the operator $\hat{\psi}^+ = \psi^+(q^{3/4})$, $\hat{b}^\pm = b^\pm(q^{3/4})$, which satisfy, respectively:

$$\hat{\psi}\hat{\psi}^+ + q^{3/2}\hat{\psi}^+\hat{\psi} = q^{3Q/2} \tag{19a}$$

$$\hat{b}\hat{b}^+ - q^{3/2}\hat{b}^+\hat{b} = q^{-3M/2} . \tag{19b}$$

Then we write $(q \neq -1)$

$$E_4 = \sqrt{\frac{q + q^{-1} + 1}{q^{1/2} + q^{-1/2}}}\,\hat{\psi}\hat{b}^+ \qquad F_4 = \sqrt{\frac{q + q^{-1} + 1}{q^{1/2} + q^{-1/2}}}\,\hat{\psi}^+\hat{b} . \tag{20}$$

Using the identity:

$$(q^{-3/2} - q^{3/2}) = (q + q^{-1} + 1)(q^{-1/2} - q^{1/2}) \tag{21}$$

we compute

$$[E_4,\, F_4] = \frac{q + q^{-1} + 1}{q^{1/2} + q^{-1/2}}\,[Q + M]_{q^{-3/2}} = \left[\frac{3}{2}(Q + M)\right]_{q^{-1}} . \tag{22}$$

It is immediate to compute that

$$[Q + M,\, E_4] = [Q + M,\, F_4] = 0 . \tag{23}$$

Using the identity:

$$[AB,\, C] = A[B,\, C] + [A,\, C]B \tag{24}$$

$$[[A,\, B],\, C] = [A,\, [B,\, C]] + [[A,\, C],\, B] \tag{25}$$

one easily finds, using equations (10*b*) and (10*c*) and $a_i^+ a_j = \frac{1}{2}[a_i^+,\, a_j]$ that $(k = 1,\, 2)$:

$$[\hat{\psi}^+\hat{\psi},\, a_k^+ a_{k+1}] = [\hat{\psi}^+\hat{\psi},\, a_{k+1}^+ a_k] = 0 . \tag{26}$$

Then from equation (11), using the above specified prescription to get the analogous equation in the deformed case, we rewrite equations (23)–(26):

$$\begin{aligned} Q &= \sum_{i=1}^{3} c_i N_i \\ \sum_{i=1}^{3} [c_i N_i, \hat{\psi}] &= -\hat{\psi} = -\sum_{i=1}^{3} c_i \frac{1}{2} \hat{\psi} \\ \sum_{i=1}^{3} [c_i N_i, a_1^+ a_2] &= 0 = c_1 - c_2 \\ \sum_{i=1}^{3} [c_i N_i, a_2^+ a_3] &= 0 = c_2 - c_3 \end{aligned} \tag{27}$$

which implies

$$c_i = 2/3 \qquad (i = 1, 2, 3). \tag{28}$$

Equation (22) can be written

$$[E_4, F_4] = [N_1 + N_2 + N_3 + \tfrac{3}{2} M]_{q^{-1}} \tag{29}$$

so

$$\begin{aligned} H_4 &= N_1 + N_2 + N_3 + \tfrac{3}{2} M \\ &= \tfrac{1}{2}(2H_1 + 4H_2 + 3H_3) + \tfrac{3}{2} M \end{aligned} \tag{30}$$

which is just the usual expression [4] for the corresponding Cartan in F(4).

It is easy to compute

$$[E_i, F_j] = 0 \qquad (i = 1, 2, 3, 4, \quad i \neq j) \tag{31}$$

and that the action of H_i on E_j^+ $(i, j = 1, 2, 3, 4)$ gives the Cartan–Kac matrix (15).

Now we have to verify the Serre relations.

For $1 \leqslant i, j \leqslant 3$ these relations are equivalent to the Serre relations for $U_q(B_3)$, so we refer to [3], even if their verification is rather immediate. So we limit ourselves to the following entries of the modified Cartan–Kac matrix $\hat{a}_{i4}$, $\hat{a}_{4i}$ $(i < 4)$. For $i = 1, 2$ the verification follows from identity equation (24) and equations (10*b*) and (10*c*), while for $i = 3$ each monomial in equation (9*b*) vanishes due to vanishing of the square of a q-fermion.

The Cartan matrix of G(3), in the distinguished basis, is given by (see [4])

$$\begin{vmatrix} 2 & -1 & 0 \\ -3 & 2 & -1 \\ 0 & 1 & 0 \end{vmatrix}. \tag{32}$$

The values of d_i are: $d_i = (3, 1, -1)$.

The generators (deg = 0) E_k (F_k) ($k = 1, 2$) give a realization of $U_q(G_2)$. In [10] a realization of this quantum algebra has been given in terms of q-quasiparafermions. Here we present a different, simpler realization in terms of q-fermions.

Let us introduce q-fermions $A_k^+ = a_k^+(q^3)$, $A_k = a_k(q^3)$ ($k = 1, 2$) which satisfy equation (12*a*) replacing q by q^3.

Then we can realize the generators of $G_q(3)$ as follows:

$$\begin{aligned}
&E_1 = A_1^+ A_2 \qquad F_1 = A_2^+ A_1 \\
&E_2 = q^{-(N_3-N_1)} \sqrt{\frac{q^{2(N_2-1)} + q^{-2(N_2-1)}}{2}}\, a_2^+ + q^{2(N_2-1)}\, a_3^+ a_1 \\
&F_2 = q^{-(N_3-N_1)}\, a_2 \sqrt{\frac{q^{2(N_2-1)} + q^{-2(N_2-1)}}{2}} + q^{2(N_2-1)}\, a_1^+ a_3
\end{aligned} \tag{33}$$

and from our definition of A_k^+, A_k ($k = 1,2$) and our conjecture we get

$$\begin{aligned}
&[E_1, F_1] = [N_1 - N_2]_{q^6} \\
&H_1 = N_1 - N_2.
\end{aligned} \tag{34}$$

A straightforward and tedious calculation gives

$$\begin{aligned}
&[E_2, F_2] = [2N_2 + N_3 - N_1]_{q^2} \\
&H_2 = 2N_2 + N_3 - N_1.
\end{aligned} \tag{35}$$

The only generator of deg = 1 can be written ($q^2 \neq -1$)

$$E_3 = \sqrt{q + q^{-1}}\, \psi b^+ \qquad F_3 = \sqrt{q + q^{-1}}\, \psi^+ b. \tag{36}$$

We have

$$[E_3, F_3] = (q + q^{-1})[Q + M]_{q^{-2}} = [2(Q + M)]_{q^{-1}}. \tag{37}$$

It is immediate to compute that:

$$[Q + M, E_3] = [Q + M, F_3] = 0. \tag{38}$$

Now in analogous way with the case of $F_q(4)$ we find:

$$\begin{aligned}
&Q = \sum_{i=1}^{3} c_i N_i \qquad \sum_{i=1}^{3} [c_i N_i, \psi] = -\psi = -\sum_{i=1}^{3} \frac{c_i}{2} \psi \\
&\sum_{i=1}^{3} [c_i N_i, E_1] = 0 = c_1 - c_2 \\
&\sum_{i=1}^{3} [c_i N_i, E_2] \\
&\qquad = c_2 \left(q^{-(N_3-N_1)} \sqrt{\frac{q^{2(N_2-1)} + q^{-2(N_2-1)}}{2}}\, a_2^+ \right) + (c_3 - c_1) q^{2(N_2-1)}\, a_3^+ a_1.
\end{aligned} \tag{39}$$

Requiring that E_2 is an eigenvector of Q implies

$$c_2 = c_3 - c_1. \tag{40}$$

So we find:

$$c_1 = c_2 = \tfrac{1}{2} \qquad c_3 = 1. \tag{41}$$

We can rewrite equation (37) in the form

$$\begin{aligned} &[E_3, F_3] = [N_1 + N_2 + 2N_3 + 2M]_{q^{-1}} \\ &H_3 = N_1 + N_2 + 2N_3 + 2M = 3H_1 + 2H_2 + 2M. \end{aligned} \tag{42}$$

Again we find the usual expression [4] of the Cartan matrix element in G(3).

It is quite easy to check that

$$[E_i, F_j] = 0 \qquad (i, j = 1, 2, 3, \quad i \neq j) \tag{43}$$

and that the action of H_1 $(i, j = 1, 2, 3)$ gives the Cartan–Kac matrix (32).

The verification of the Serre relations is a easy task as each monomial in equations (7)–(9*b*) for $\hat{a}_{ij} \neq 0$ is vanishing and for $\hat{a}_{ij} = 0$ we have already shown that they are satisfied in $F_q(4)$ and the structure of the relevant generator E_1 in both the q-superalgebras is (up to a re-scaling in q, irrelevant in this context) the same.

An essential point in our realization of $F_q(4)$ and $G_q(3)$ has been the conjecture that the q-number fermionic operator does not depend on q. This conjecture is, really, an essential point in the realization of the even part of $G_q(3)$, that is $U_q(G_2)$, while its use in the construction of $F_q(4)$ has allowed to find the usual form of H_4. However, it is well known that Cartan matrix are defined up to the multiplication for an arbitrary (non-degenerate) matrix. So the rescaling in q of the q-generators is the analogue of the multiplication by a c-factor of the generators in the case of SLAS.

It would be interesting to have an explicit construction of q-fermions to be able to verify this conjecture.

Finally let us remark that the construction of the exceptional q-superalgebra is simpler than the construction of the non-deformed corresponding ones which would require the introduction of more fermions and more relations beside equation (10) in order to be able to compute the supercommutator of the generators.

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